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THE PROBLEM OF RELATIVE MAXIMA OR MINIMA UNDER A NEW POINT OF VIEW.*

By MR. CHAS. H. KUMMELL, Washington, D. C.

Price, in his Infinitesimal Calculus, Vol. I, pp. 235 to 278, gives a treatment of this problem, complete in every respect save one; for on page 274 he dismisses the problem of discriminating between a maximum or minimum, in case of a relative maximum or minimum, by saying: "The algebraical criterion for discriminating between a maximum and a minimum is too complicated to be of any service even in this particular case."† While admitting this difficulty, the reader should however, not be left in any doubt as to the method to be followed, if, in spite of this difficulty, he wishes to make the discrimination. Since Price gives a very elaborate treatment of this discrimination problem for a function of n variables we may of course, by eliminating m variables by m conditions, and thus forming a function of $n - m$ variables, apply his methods to this latter. This evasion of the method of relative maxima or minima is usually highly inelegant, and there should be a discriminating method immediately applicable.

Suppose we require the conditions for maximum or minimum of the function

$$F(x_1, x_2, \ldots x_n), \tag{1}$$

subject to the m conditions

$$\begin{aligned} f_1\,(x_1, x_2, \ldots x_n) &= 0, \\ f_2\,(x_1, x_2, \ldots x_n) &= 0, \\ \cdots\cdots& \\ f_m(x_1, x_2, \ldots x_n) &= 0. \end{aligned} \tag{2}$$

If we form the new function

$$u = F + c_1 f_1 + c_2 f_2 + \ldots + c_m f_m, \tag{3}$$

it is at once apparent that, by virtue of (2), u is identical with F, although it has

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*Read before the Mathematical Section of the Philosophical Society of Washington.

†It is the case of but one condition between the variables.

$n + m$ variables. If we subject this new function u to the conditions of a maximum or minimum, we have

$$0 = \frac{du}{dx_1} = \frac{du}{dx_2} = \ldots = \frac{du}{dx_n}, \tag{4}$$

$$\begin{aligned} 0 &= \frac{du}{dc_1} = f_1, \\ 0 &= \frac{du}{dc_2} = f_2, \\ &\cdots\cdots \\ 0 &= \frac{du}{dc_m} = f_m; \end{aligned} \tag{2'}$$

and since the latter m conditions are identical with the given conditions (2), it follows that the function u *need not be considered subject to any conditions.* In using the function u in place of F we have therefore the problem to find the conditions of maximum or minimum of a function of $n + m$ variables, and since Price has given, pp. 252–265, a complete treatment of the problem of discriminating between a maximum or a minimum of a function of any number of variables which are not subject to any condition, we have thus at the same time the required directly applicable method for relative maxima and minima.

To apply these principles to an easy problem, let it be required to divide a given number a into two parts x and y so that $x^m y^n$ shall be a maximum (or minimum?). We may take for F in this case the logarithm of $x^m y^n$; and since

$$x + y - a = 0,$$

we have

$$u = mlx + nly + c(a - x - y);$$

$$\begin{aligned} 0 &= \frac{du}{dx} = \frac{m}{x} - c, & \frac{d^2u}{dx^2} &= -\frac{m}{x^2}, & \frac{d^2u}{dydc} &= -1; \\ 0 &= \frac{du}{dy} = \frac{n}{y} - c, & \frac{d^2u}{dy^2} &= -\frac{n}{y^2}, & \frac{d^2u}{dcdx} &= -1; \\ 0 &= \frac{du}{dc} = a - x - y, & \frac{d^2u}{dc^2} &= 0, & \frac{d^2u}{dxdy} &= 0. \end{aligned}$$

We obtain

$$x = \frac{ma}{m+n}, \quad y = \frac{na}{m+n};$$

and we have to decide whether $\left(\frac{ma}{m+n}\right)^m \left(\frac{na}{m+n}\right)^n$, or else $m^m n^n \left(\frac{a}{m+n}\right)^{m+n}$, is a maximum or a minimum. To form the discriminating cubic we have

$$\begin{aligned}\left(\frac{m}{x^2}+\theta\right)dx &+ \quad 0\,.\,dy &+ \quad dc &= 0,\\ 0\,.\,dx &+ \left(\frac{n}{y^2}+\theta\right)dy &+ \quad dc &= 0,\\ dx &+ \quad dy &+ (0+\theta)\,dc &= 0;\end{aligned}$$

whence
$$0=\begin{vmatrix}\frac{m}{x^2}+\theta, & 0, & 1\\ 0, & \frac{n}{y^2}+\theta, & 1\\ 1, & 1, & \theta\end{vmatrix}$$

$$=\left(\frac{m}{x^2}+\theta\right)\left(\frac{n}{y^2}+\theta\right)\theta-\left(\frac{m}{x^2}+\theta\right)-\left(\frac{n}{y^2}+\theta\right)$$

$$=\theta^3+\left(\frac{m}{x^2}+\frac{n}{y^2}\right)\theta^2+\left(\frac{mn}{x^2y^2}-2\right)\theta-\frac{m}{x^2}-\frac{n}{y^2}.$$

Because the 2d and 4th terms are always of opposite sign, it will be impossible that the three roots of this discriminating cubic can be of the same sign; consequently there can be no total maximum with respect to the three variables x, y, c. Leaving the third variable c out of consideration, we have for a partial maximum or minimum the discriminating quadratic

$$\left(\frac{m}{x^2}+\theta\right)\left(\frac{n}{y^2}+\theta\right)=0=\theta^2+\left(\frac{m}{x^2}+\frac{n}{y^2}\right)\theta+\frac{mn}{x^2y^2}.$$

The roots of this equation are always negative if m and n are positive; hence the condition for a partial maximum with respect to x and y are fulfilled, since then likewise d^2u/dx^2 and d^2u/dy^2 are negative.

As in this case, so in others, it will be unnecessary to consider the total maximum or minimum, but only the partial maximum or minimum for the given variables.